UNIVERSITY
OF WARSAW

Faculty of Mathematics, Informatics and Mechanics
Dean Professor Paweł Strzelecki

Dean of the Faculty of Mathematics, Informatics and Mechanics
University of Warsaw

announces
the public defense of the doctoral thesis

Title of thesis:

Functional and transport inequalities and their applications to concentration of measure

Candidate:

mgr Michał Strzelecki

The defense will take place on October 3rd, 2019 at 14:30
in room **2180** (**1st** floor)
at the Faculty of Mathematics, Informatics and Mechanics
of the University of Warsaw, Banacha 2

Supervisor: dr hab. Radosław Adamczak (University of Warsaw)

Referees: dr hab. Mateusz Kwaśnicki (Wroclaw University of Science and Technology)
prof. Paul-Marie Samson (University Paris-Est Marne-La-Vallée)

Dissertation, summary of PhD dissertation and reviews
are available at www.mimuw.edu.pl/doktoraty/michal-strzelecki

Address: ul. Banacha 2; 02–097 Warszawa; Poland
Phone (+48 22) 554 42 14
e-mail: dean@mimuw.edu.pl web page: http://www.mimuw.edu.pl/

Author's abstract of the PhD dissertation

Functional and transport inequalities and their applications to concentration of measure

Michał Strzelecki

1. Introduction and motivation

My thesis is devoted to the study of functional and transportation inequalities connected to the concentration of measure phenomenon. Before presenting its results let us start with a short historical introduction.

The solution to the classical isoperimetric problem on the sphere $S^{n-1} \subset \mathbb{R}^n$ has the following striking consequence: functions on high-dimensional spheres with small local oscillations are essentially constant (they are concentrated around their median (or mean value)). The importance of this elementary, yet non-trivial, observation was emphasized by Milman, who used it in his proof of Dvoretzky's theorem [**23**]. The concentration of measure phenomenon has become one of the main themes of high dimensional probability and geometric analysis (see, e.g., the monographs [**21, 8**]).

A substantial body of research has been devoted to the interplay between various functional inequalities, transportation of measure theory, and the concentration of measure phenomenon, showing intimate connections between them (to mention, e.g., [**26, 22, 20, 6, 24, 5, 10, 11, 13**]). While initial results on concentration of measure concerned mostly deviation bounds for Lipschitz functions of highly regular random variables, the work by Talagrand [**26, 27**] has revealed that if one restricts attention to convex Lipschitz functions, dimension-free concentration of measure holds under much weaker conditions.

Even though the theory of concentration of measure for convex functions to some extent parallels the classical theory, there are some subtle differences related to the fact that convexity is not preserved even under the change of signs. This creates certain difficulties in the proofs and invalidates many well known arguments from the classical setting. Nevertheless, several important results have been obtained, including the characterization of dimension-free convex concentration in terms of the convex Poincaré inequality [**13**] and the recent work on weak transport–entropy inequalities [**15, 14**].

The research presented in my thesis is strictly connected to the last two articles. It originated from the following question.

Question 1.1. *Can one give a sufficient and necessary condition for a probability measure on the real line to satisfy the log-Sobolev inequality restricted to the the class of convex functions?*

I shall discuss the results obtained in my thesis (which include, but are not limited to, the answer to the above question) in the next section.

2010 *Mathematics Subject Classification.* Primary: 60E15. Secondary: 26A51, 26B25, 26D10.

Key words and phrases. Concentration of measure, convex functions, infimum convolution, log-Sobolev inequality, Poincaré inequality, transportation inequalities.

2. Results

2.1. Organization of the thesis and articles it is based on. My thesis is based mainly on the following three articles:

1. M. Strzelecka, M. Strzelecki, T. Tkocz, *On the convex infimum convolution inequality with optimal cost function*, ALEA Lat. Am. J. Probab. Math. Stat. 14 (2017), 903–915,
2. Y. Shu, M. Strzelecki, *A characterization of a class of convex log-Sobolev inequalities on the real line*, Ann. Inst. Henri Poincaré Probab. Stat. 54 (2018), 2075–2091,
3. R. Adamczak, M. Strzelecki, *On the convex Poincaré inequality and weak transportation inequalities*, accepted in Bernoulli,

and on results obtained in spring 2018, while I was visiting the Institute of Mathematics in Toulouse, France, and working with Franck Barthe (they are part of a project which at the time of writing is still in progress).

The organization of the thesis is the following. In the first part, we work in the classical setting of smooth functions and are interested in the concentration between the exponential and Gaussian levels: Chapter 1 is a general introduction and in Chapter 2 we present a new result about Beckner-type inequalities of Latała and Oleszkiewicz.

The second, more extensive, part is concerned with concentration of measure for convex functions. The main tool, used throughout, is the theory of weak transportation inequalities introduced recently by Gozlan, Roberto, Samson, and Tetali [**15**].

We start with some preliminaries (Chapter 3), then study the convex log-Sobolev inequality on the real line (Chapter 4), the convex Poincaré inequality in $\mathbb{R}^n$ (Chapter 5), general refined concentration of measure inequalities, which are consequences of weak transportation inequalities (Chapter 6), and convex infimum convolution inequalities with optimal cost functions for measures with log-concave tails (Chapter 7).

Below I describe the most important results contained in my thesis and some of their consequences.

2.2. Beckner-type inequalities and modified log-Sobolev inequalities. Fix $r \in (1,2)$ and $q \in (2,\infty)$ such that $1/r + 1/q = 1$. In order to prove concentration results for (the products) of measures which have heavier tails than the standard Gaussian measure various variants of the log-Sobolev inequality have been introduced in the literature, among others Beckner-type inequalities of Latała and Oleszkiewicz [**16**] and modified log-Sobolev inequalities of Gentil, Guillin, and Miclo [**10**].

We say that a probability measure μ on $\mathbb{R}^d$ satisfies the *Latała–Oleszkiewicz inequality* if there exists a constant $C_{LO} < \infty$ such that for every smooth $f\colon \mathbb{R}^d \to \mathbb{R}$,

$$\sup_{p\in(1,2)} \frac{\int_{\mathbb{R}^d} f^2 d\mu - \Big(\int_{\mathbb{R}^d} |f|^p d\mu\Big)^{2/p}}{(2-p)^{2(1-1/r)}} \le C_{LO} \int_{\mathbb{R}^d} |\nabla f|^2 d\mu. \tag{2.1}$$

We say that a probability measure μ on $\mathbb{R}^d$ satisfies the *modified log-Sobolev inequality* if there exists a constant $C_{mLS} < \infty$ such that for every smooth function $f\colon \mathbb{R}^d \to (0,\infty)$,

$$\mathrm{Ent}_\mu(f^2) \le C_{mLS} \int_{\mathbb{R}^d} H_q\Big(\frac{|\nabla f|}{f}\Big) f^2 d\mu, \tag{2.2}$$

where $H_q(t) := \max\{t^2, |t|^q\}$ for $t \in \mathbb{R}$ (recall that $q = r/(r-1)$).

Both those inequalities have the tensorization property, but the concentration properties implied by (2.2) are—up to constants—better than those implied by (2.1).[1] On the other hand, it is natural to conjecture, that—as in the result of Bobkov and Ledoux [**4**] concerning the Poincaré inequality—the Latała–Oleszkiewicz inequality implies the modified log-Sobolev inequality (and therefore improved two-level concentration).[2] The main result of Chapter 2 of my PhD thesis states that this is indeed the case:

THEOREM 2.1. *Let μ be a probability measure on $\mathbb{R}^d$ which satisfies the Latała–Oleszkiewicz inequality* (2.1) *with constant C_{LO}. Then μ satisfies the modified log-Sobolev inequality* (2.2) *with a constant C_{mLS} depending only on C_{LO} and r.*

Theorem 2.1 implies an improvement of the concentration properties which were derived from (2.1) by Latała and Oleszkiewicz [**16**] and Gozlan [**12**]. In particular, if μ is a probability measure on $\mathbb{R}^d$ which satisfies inequality (2.1) with constant C_{LO}, then there exists a constant $K = K(C_{LO}, r) > 0$, such that for any $n \in \mathbb{N}$ and any set $A \subset \mathbb{R}^{dn}$ with $\mu^{\otimes n}(A) \geq 1/2$,

$$\mu^{\otimes n}\big(A + \sqrt{t}B_2^{dn} + t^{1/r}B_{r,2}^{n,d}\big) \geq 1 - e^{-Kt}.$$

Here B_2^{dn} stands for the unit balls in the ℓ_2-norm on $\mathbb{R}^{dn}$ and

$$B_{r,2}^{n,d} := \Big\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : \big(\sum_{i=1}^n |x_i|^r\big)^{1/r} \leq 1\Big\}.$$

In Chapter 2 of my thesis I also discuss connections between modified log-Sobolev inequalities and weighted log-Sobolev inequalities.

2.3. Characterization of the convex log-Sobolev inequality the real line. Let μ be a Borel probability measure on $\mathbb{R}$. We say that μ satisfies the *convex log-Sobolev inequality* if for every smooth convex Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$,

$$\mathrm{Ent}_\mu(e^f) \leq C \int_{\mathbb{R}} |f'|^2 e^f \, d\mu. \tag{2.3}$$

Let U_μ be the unique left-continuous and non-decreasing map transporting the symmetric exponential measure onto the reference measure μ,

$$U_\mu(x) = F_\mu^{-1} \circ F_\tau(x) = \begin{cases} F_\mu^{-1}(\frac{1}{2}e^{-|x|}) & \text{if } x < 0, \\ F_\mu^{-1}(1 - \frac{1}{2}e^{-|x|}) & \text{if } x \geq 0. \end{cases}$$

Here F_μ^{-1} is the generalized inverse of the cumulative distribution function F_μ.

The following strengthening of the results obtained recently in [**14**] is the main result of Chapter 4 of my thesis. (A similar result holds true also for convex modified log-Sobolev inequalities.)

[1] In particular, one can recover the tail behavior from the Central Limit Theorem.

[2] In the case when μ is a probability measure on the real line one can prove criteria for the Latała–Oleszkiewicz inequality (2.1) and the modified log-Sobolev inequality (2.2), see [**2, 3**]. To the best of our knowledge, it does not seem to be possible to find a simple argument which would allow us to deduce the modified log-Sobolev inequality from the Latała–Oleszkiewicz inequality (for measures on the real line) at the level of those characterizations.

THEOREM 2.2. *The following conditions are equivalent.*

(i) For every $s > 0$ we have $\int_{\mathbb{R}} e^{s|x|} d\mu(x) < \infty$ and there exists $C > 0$ such that μ satisfies the convex log-Sobolev inequality (2.3)*.*

(ii) There exist $a, b > 0$ such that for all $h > 0$,

$$\sup_{x\in\mathbb{R}}\{U_\mu(x+h) - U_\mu(x)\} \leq \sqrt{a + bh}.$$

In each of the implications the constants in the conclusion depend only on the constants in the premise.

The above theorem is stated without referring to weak transport–entropy inequalities, but they lie at the heart of the proof and in fact Theorem 2.2 shades new light on the relationship between the inequalities $\overline{\mathbf{T}}_\theta$ and $\overline{\mathbf{T}}_\theta^-$.

As an important consequence, we obtain concentration bounds for both the upper and lower tails of convex Lipschitz functions. Let us stress here that for the lower tail such estimates were previously unknown since working directly with the convex log-Sobolev inequality leads only to bounds for the upper tail, see [**20**].

2.4. Convex Poincaré inequality in $\mathbb{R}^n$. We say that a probability measure μ on $\mathbb{R}^n$ satisfies the *convex Poincaré inequality* with constant $\lambda > 0$ if for all convex functions $f\colon \mathbb{R}^n \to \mathbb{R}$ we have

$$\operatorname{Var} f(X) \leq \frac{1}{\lambda}\,\mathbb{E}\,|\nabla f(X)|^2, \tag{2.4}$$

where X is a random vector with law μ and by $|\nabla f(x)|$ we mean the length of gradient at x, defined as

$$|\nabla f(x)| = \limsup_{y\to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

The main result of Chapter 5 is a convex counterpart of the famous result of Bobkov and Ledoux [**4**] which concerns the classical Poincaré inequality for smooth functions. In the convex setting, such a result was obtained recently independently in [**9**] and [**14**], but only for measures on the real line.

THEOREM 2.3. *Let μ be a probability measure on $\mathbb{R}^n$ which satisfies the convex Poincaré inequality* (2.4)*; let X be a random vector with law μ. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex with $|\nabla f(x)| \leq c < \sqrt{2\lambda}/(32e)$ for all $x \in \mathbb{R}^n$, then*

$$\operatorname{Ent}(e^{f(X)}) \leq C(c,\lambda)\,\mathbb{E}\,|\nabla f(X)|^2 e^{f(X)},$$
$$\operatorname{Ent}(e^{-f(X)}) \leq C(\lambda,c,M)\,\mathbb{E}\,|\nabla f(X)|^2 e^{-f(X)},$$

where $M \in \mathbb{R}_+$ is any number that satisfies $\mathbb{P}(|X - \mathbb{E}\,X| \leq M) \geq 3/4$.

REMARK 2.4. We suspect that the dependence on M is an artifact of our proof.

Theorem 2.3 implies in particular that μ satisfies the convex Poincaré inequality if and only if it satisfies the weak transport-entropy inequality $\overline{\mathbf{T}}$ with a quadratic linear cost function.

We also present refined concentration of measure inequalities, which are consequences of weak transportation inequalities (or, equivalently, their dual formulations: convex infimum convolution inequalities). This includes applications to concentration for non-Lipschitz convex functions in the spirit of recent results due to Bobkov, Nayar, and Tetali [**7**].

2.5. Convex infimum convolution inequalities with optimal cost functions. Let X be a random vector with values in $\mathbb{R}^n$ and let $\varphi : \mathbb{R}^n \to [0, \infty]$ be a measurable function. We say that the pair (X, φ) satisfies the *convex infimum convolution inequality* if for every convex function $f \colon \mathbb{R}^n \to \mathbb{R}$ bounded from below,

$$\mathbb{E}\, e^{f \Box \varphi(X)}\, \mathbb{E}\, e^{-f(X)} \leq 1, \tag{2.5}$$

where $f \Box \varphi$ denotes the infimum convolution of f and φ defined as

$$f \Box \varphi(x) = \inf\{f(y) + \varphi(x - y) : y \in \mathbb{R}^n\}, \quad x \in \mathbb{R}^n.$$

For a random vector X in $\mathbb{R}^n$ define

$$\Lambda_X^*(x) := \mathcal{L}\Lambda_X(x) := \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - \ln \mathbb{E}\, e^{\langle y, X\rangle}\}.$$

If X is symmetric and the pair (X, φ) satisfies the convex infimum convolution inequality, then $\varphi(x) \leq \Lambda_X^*(x)$ for every $x \in \mathbb{R}^n$ (see Remark 2.12 in [**19**]). In other words, Λ_X^* is the optimal cost function φ for which the convex infimum convolution inequality can hold.

In Chapter 7 we apply the results of [**14**] to get the following result.

Theorem 2.5. *Let X be a symmetric random variable with log-concave tails, i.e., such that the function*

$$t \mapsto N(t) := -\ln \mathbb{P}(|X| \geq t), \quad t \geq 0,$$

is convex. Then there exists a universal constant $\beta \leq 1680e$ such $(X, \Lambda_X^*(\cdot/\beta))$ satisfies the convex infimum convolution inequality.

As a consequence, one obtains the comparison of weak and strong moments of random vectors with independent coordinates with log-concave tails in the spirit of the results from [**25, 1, 18, 17**].

3. Other publications and preprints

I am also a (co-)author of the following articles and preprints, which are not directly connected to the subject of my PhD thesis:

4. R. Adamczak, M. Strzelecki, *Modified log-Sobolev inequalities for convex functions on the real line. Sufficient conditions*, Studia Math. 230 (2015), 59–93,
5. M. Strzelecki, *A note on sharp one-sided bounds for the Hilbert transform*, Proc. Amer. Math. Soc. 144 (2016), 1171–1181,
6. M. Strzelecki, *The L^p-norms of the Beurling–Ahlfors transform on radial functions*, Ann. Acad. Sci. Fenn. Math. 42 (2017), 73–93,
7. R. Adamczak, M. Kotowski, B. Polaczyk, M. Strzelecki, *A note on concentration for polynomials in the Ising model*, arXiv e-prints.

The article 4. contains some preliminary results which were superseded by [**14**] and the article 2. listed above. The articles 5., 6. are concerned with martingale inequalities and their applications to harmonic analysis. The article 7. is more directly connected to the subject of my PhD thesis: we use some results obtained in the article 3. listed above to obtain concentration bounds for quadratic forms in bounded, dependent random variables (under some assumptions which are fulfilled, e.g., in the Ising model satisfying the Dobrushin condition).

References

1. R. Adamczak, R. Latała, A. E. Litvak, K. Oleszkiewicz, A. Pajor, and N. Tomczak-Jaegermann, *A short proof of Paouris' inequality*, Canad. Math. Bull. **57** (2014), no. 1, 3–8. MR 3150710 5
2. F. Barthe and C. Roberto, *Sobolev inequalities for probability measures on the real line*, Studia Math. **159** (2003), no. 3, 481–497, Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday (Polish). MR 2052235 3
3. ______, *Modified logarithmic Sobolev inequalities on* $\mathbb{R}$, Potential Anal. **29** (2008), no. 2, 167–193. MR 2430612 3
4. S. Bobkov and M. Ledoux, *Poincaré's inequalities and Talagrand's concentration phenomenon for the exponential distribution*, Probab. Theory Related Fields **107** (1997), no. 3, 383–400. MR 1440138 3, 4
5. S.G. Bobkov, I. Gentil, and M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. (9) **80** (2001), no. 7, 669–696. MR 1846020 1
6. S.G. Bobkov and F. Götze, *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal. **163** (1999), no. 1, 1–28. MR 1682772 1
7. S.G. Bobkov, P. Nayar, and P. Tetali, *Concentration properties of restricted measures with applications to non-Lipschitz functions*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2169, Springer, Cham, 2017, pp. 25–53. MR 3645113 5
8. S. Boucheron, G. Lugosi, and P. Massart, *Concentration inequalities*, Oxford University Press, Oxford, 2013, A nonasymptotic theory of independence, With a foreword by Michel Ledoux. MR 3185193 1
9. N. Feldheim, A. Marsiglietti, P. Nayar, and J. Wang, *A note on the convex infimum convolution inequality*, Bernoulli **24** (2018), no. 1, 257–270. MR 3706756 4
10. I. Gentil, A. Guillin, and L. Miclo, *Modified logarithmic Sobolev inequalities and transportation inequalities*, Probab. Theory Related Fields **133** (2005), no. 3, 409–436. MR 2198019 1, 2
11. N. Gozlan, *A characterization of dimension free concentration in terms of transportation inequalities*, Ann. Probab. **37** (2009), no. 6, 2480–2498. MR 2573565 1
12. ______, *Poincaré inequalities and dimension free concentration of measure*, Ann. Inst. Henri Poincaré Probab. Stat. **46** (2010), no. 3, 708–739. MR 2682264 3
13. N. Gozlan, C. Roberto, and P.-M. Samson, *From dimension free concentration to the Poincaré inequality*, Calc. Var. Partial Differential Equations **52** (2015), no. 3-4, 899–925. MR 3311918 1
14. N. Gozlan, C. Roberto, P.-M. Samson, Y. Shu, and P. Tetali, *Characterization of a class of weak transport-entropy inequalities on the line*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 3, 1667–1693. MR 3825894 1, 3, 4, 5
15. N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali, *Kantorovich duality for general transport costs and applications*, J. Funct. Anal. **273** (2017), no. 11, 3327–3405. MR 3706606 1, 2
16. R. Latała and K. Oleszkiewicz, *Between Sobolev and Poincaré*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1745, Springer, Berlin, 2000, pp. 147–168. MR 1796718 2, 3
17. R. Latała and M. Strzelecka, *Weak and strong moments of ℓ_r-norms of log-concave vectors*, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3597–3608. MR 3503729 5
18. ______, *Comparison of weak and strong moments for vectors with independent coordinates*, Mathematika **64** (2018), no. 1, 211–229. MR 3778221 5
19. R. Latała and J. O. Wojtaszczyk, *On the infimum convolution inequality*, Studia Math. **189** (2008), no. 2, 147–187. MR 2449135 5
20. M. Ledoux, *On Talagrand's deviation inequalities for product measures*, ESAIM Probab. Statist. **1** (1995/97), 63–87 (electronic). MR 1399224 (97j:60005) 1, 4
21. ______, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001. MR 1849347 (2003k:28019) 1
22. B. Maurey, *Some deviation inequalities*, Geom. Funct. Anal. **1** (1991), no. 2, 188–197. MR 1097258 1

23. V. D. Milman, *A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies*, Funkcional. Anal. i Priložen. **5** (1971), no. 4, 28–37. 1
24. F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173** (2000), no. 2, 361–400. MR 1760620 1
25. G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. **16** (2006), no. 5, 1021–1049. MR 2276533 5
26. M. Talagrand, *A new isoperimetric inequality and the concentration of measure phenomenon*, Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., vol. 1469, Springer, Berlin, 1991, pp. 94–124. MR 1122615 (93d:60095) 1
27. ———, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 81, 73–205. MR 1361756 (97h:60016) 1